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Infinite Monochromatic Subgraphs

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1 Introduction

Proofs are not given here. See [5] for the details. First we recall the following famous theorems of Ramsey:

- Finite Ramsey Theorem (FRT)

$$\forall x, y, z \in \omega \exists z \in \omega[z \rightarrow (x)_z^y].$$

- Infinite Ramsey Theorem (IRT)

$$\forall x, y \in \omega[\omega \rightarrow (\omega)_y^x].$$

To explain structural Ramsey theory, we introduce the notation $\binom{B}{A}$, where A and B are first order structures.

$$\binom{B}{A} = \text{the set of all copies of } A \text{ in } B.$$

Then, FRT for $y = 2$ ($\forall x, y, z \in \omega \exists z \in \omega[z \rightarrow (x)_z^2]$) is equivalent to the following statement:

(*) For all complete finite graphs A and $z \in \omega$ there is a complete finite graph D such that if $f : [D]^2 \rightarrow z$ is a finite coloring of the edges in D then there is $A' \in \binom{D}{A}$ for which $f([A']^2)$ is a singleton.

In addition to this, IRT for $x = 2$ ($\forall y \in \omega[\omega \rightarrow (\omega)_y^2]$) is equivalent to

(**) For every infinite complete graph G and a finite edge coloring $f : [G]^2 \rightarrow y$ there is an infinite complete subgraph $H \subset G$ such that $f([H]^2)$ is a singleton.

So the classical Ramsey theorem (finite version or infinite version) trivially provides results on edge colorings of complete graphs. Structural Ramsey theory studies Ramsey type results on more general structures other than complete graphs.

2 Ramsey Class

Let L be a relational language and K a class of finite L-structures. We assume K satisfies the conditions of Fraïssé so that K has the Fraïssé limit $\mathcal{M}$.

Definition 1 (Ramsey Class)**.** K is a Ramsey class if

- $\forall A, B \in K$, $\forall n \in \omega$, $\exists C \in K$ s.t. for every n-coloring
$$f : \binom{C}{A} \to n$$
there is $B' \in \binom{C}{B}$ for which $\binom{B'}{A}$ is monochromatic.

Example 2. In a sense, a Ramsey class is a class in which FRT holds. The following classes are examples of Ramsey classes.

- K_1 = the class of linearly ordered finite sets. The limit $\mathcal{M}$ is isomorphic to $\mathbb{Q}$.
- K_2 = the class of linearly ordered finite (hyper)graphs. $\mathcal{M}$ is the ordered random graph.
- K_3 = the class of linearly ordered triangle-free finite graphs.

The fact that K_1 is a Ramsey class follows from FRT. Proofs for K_2 and K_3 are found in [1], [2] or [3].

Now we consider infinite versions of the above examples. We want to prove statements like: For all $n \in \omega$ and $A \in K$,

$$\mathcal{M} \to (\mathcal{M})^A_n,$$

where $\mathcal{M}$ is the limit of K. In words, this arrow statement states that if every substructure of M, isomorphic to A, is painted in one of the colors $\{0, \ldots, n-$

1}, then there is a substructure $M' \cong M$ such that every substructure of M', isomorphic to A, is pained in the same color. For $K = K_1$ (linear orders), this type of infinite version is true. In fact, it is a paraphrase of IRT. However, this version is not true in general, even if K is Ramsey.

Example 3. Let K be the class of all finite linearly ordered graphs. (This K is K_2 in the example 2.) Then the limit $\mathcal{M}$ is a linearly ordered random graph. Let $\{a_i : i \in \omega\}$ be an enumeration of $\mathcal{M}$ and let $c : E(\mathcal{M}) \to 2$ be an edge coloring defined by: for edges $\{a_i, a_j\}$,

$$c(\{a_i, a_j\}) = \begin{cases} 1 & a_i <^{\mathcal{M}} a_j \text{ and } i < j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{M}$ has no infinite substructure $D \cong \mathcal{M}$ that is homogeneous for c. To see this, let $D \subset \mathcal{M}$ be infinite. Then there must be $i < j < k \in \omega$ such that $a_i, a_j, a_k \in D$ and $\mathcal{M} \models a_i < a_k < a_j$. Then $c(\{a_i, a_j\}) = 1$ while $c(\{a_k, a_j\}) = 0$.

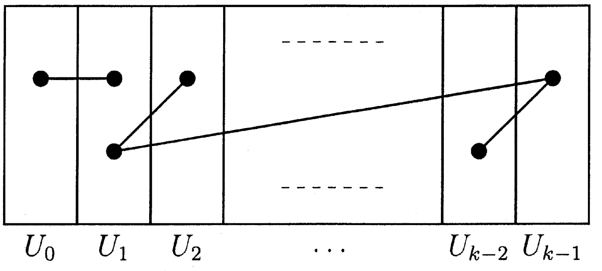
Remark 4 (An easy argument using compactness). The following weak version of RT clearly holds.

- Let K be a Ramsey class. Let M be the Fraïssé limit of K. Then for each $A, B \in K$ and each finite coloring $f : \binom{M}{A} \to n$, there is $B' \in \binom{M}{B}$ such that $\binom{B'}{A}$ is monochromatic.

3 Infinite Version

We work on k-partite graphs, where k is finite. Let $L = \{R(*,*)\} \cup \{U_i(*)\}_{i<k}$. A k-partite graph is an L-structure M such that

1. the universe of M is the disjoint union of U_i^M $(i < k)$;
2. R^M is the set of all edges in M;
3. there is no edge between two elements in the same part. $M \models \bigwedge_{i<k} \forall x, y (U_i(x) \wedge U_i(y) \to \neg R(x, y))$.



Let K be the class of all finite k-partite graphs. It has the limit $\mathcal{M}$, called a k-partite random graph.

Theorem 5. *Let $\mathcal{M}$ be a k-partite (triangle-free) random graph and f be a finite coloring on the edges. There is a k-partite-induced subgraph $N \cong \mathcal{M}$ such that f is partwise almost constant on N in the following sense: For each $a \in N$ there is a finite subset $X \subset N$ such that*

$$ax \cong_L ay \Rightarrow f(ax) = f(ay).$$

From this we can easily deduce the following famous results:

Corollary 6 (Nešetřil - Rödl). *Let K be the set of all totally ordered finite graphs. For any $B \in K$ there is $C \in K$ such that for any finite edge-coloring c on C there is $B' \in \binom{C}{B}$ such that c is constant on B.*

Corollary 7 (Nešetřil). *Let K be the set of all totally ordered triangle-free finite graphs. For any $B \in K$ there is $C \in K$ such that for any finite edge-coloring c on C there is $B' \in \binom{C}{B}$ such that c is constant on B.*

References

[1] Jaroslav Nešetřil and Vojtěch Rödl, Partitions of finite relational and set systems, Journal of Combinatorial Theory, Series A Volume 22, Issue 3, 1977, 289-312

[2] Fred G. Abramson and Leo A. Harrington, Models Without Indiscernibles, J. Symbolic Logic Volume 43, Issue 3 (1978), 572-600.

[3] Slawomir Solecki, Direct Ramsey theorem for structures involving relations and functions, J. Comb. Theory, Ser. A 01/2012; 119:440-449.

[4] Saharon Shelah, http://arxiv.org/abs/math/0504197

[5] Kota Takeuchi and Akito Tsuboi, Infnite subgraphs with monochromatic edges, submitted.

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